

On the Tractability of Un/Satisfiability

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Abstract

The Petri net approach proves to be effective to tackle the \mathcal{P} vs \mathcal{NP} problem. A safe acyclic Petri net (PN) is associated with an Exactly-1 3SAT formula, in which a clause is an exactly-1 disjunction $\dot{\vee}$ of literals. A clause also corresponds to a set of conflicting transitions in the PN. Some 2SAT/XOR-SAT formula arisen in the *inversed* PN checks if the truth assignment of a literal (a transition firing) z_v is “incompatible” for the satisfiability of the 3SAT formula (the reachability of the target state in the *inversed* PN). If z_v is incompatible, then z_v is discarded and \bar{z}_v becomes true. Therefore, a clause $(\bar{z}_v \dot{\vee} z_i \dot{\vee} z_j)$ *reduces* to the conjunction $(\bar{z}_v \wedge \bar{z}_i \wedge \bar{z}_j)$, and a 3-literal clause $(z_v \dot{\vee} z_u \dot{\vee} z_x)$ *reduces* to the 2-literal clause $(z_u \oplus z_x)$. This reduction facilitates checking un/satisfiability; the 3SAT formula is un/satisfiable iff the target state of the inversed PN is un/reachable. The solution complexity is $O(n^5)$. Therefore, it is the case that $\mathcal{P} = \mathcal{NP} = \text{co}\mathcal{NP}$.

1 Introduction

It is well known that if an \mathcal{NP} -complete problem is tractable, then all \mathcal{NP} -complete problems are tractable, i.e., $\mathcal{P} = \mathcal{NP}$, which is called the \mathcal{P} vs \mathcal{NP} problem. It is also well known that there are various formulations to specify an \mathcal{NP} -complete problem, e.g., the traveling salesman formulation. On the other hand, even if every formulation has the same solution efficiency, some formulation can be more *effective*, i.e., *facilitate finding* the solution.

This paper shows that reachability in safe acyclic Petri nets (PNs) brings about effective formulation to attack the \mathcal{P} vs \mathcal{NP} problem. A safe acyclic PN is associated with some Exactly-1 3SAT formula, and its reachability problem is \mathcal{NP} -complete [1][2][4]. This effective formulation takes place due to the *inverse* of the PN, and due to the set of *conflicts*, which specifies exactly-1 disjunction. In other words, un/reachability in the *inversed* safe acyclic PN is *tractable*, compared to un/reachability in the safe acyclic PN, because some 2SAT/XOR-SAT formula arises in the *inversed* PN that *efficiently* checks “incompatibility” of a truth assignment, and because some clauses *reduce* to conjunctions due to the incompatible assignments to be *discarded*.

The truth assignment of a literal z_i in the 3SAT formula corresponds to its associated transition occurrence (or firing) in the safe acyclic PN. In the inversed PN, a three-literal clause \mathbf{c}_k *satisfied* is denoted by a *marked* place, $\mathbf{c}_k \in M^0$, with three output transitions, each of which is a variable x_{ik} or its negation \bar{x}_{ik} in the clause \mathbf{c}_k , while a literal z_i is denoted by a place ℓ_i with two input transitions, one for x_i and the other for \bar{x}_i , i.e., $z_i \in \bullet \ell_i = \{x_i, \bar{x}_i\}$. A clause $\mathbf{c}_k = (z_i \dot{\vee} z_j \dot{\vee} z_u)$ is an exactly-1 disjunction $\dot{\vee}$, rather than disjunction \vee , of three literals to specify some Exactly-1 3SAT formula $\phi = \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \dots \wedge \mathbf{c}_m$. Then, \mathbf{c}_k is true iff either z_i or z_j or z_u is true. The initial marking (state) M^0 in the inversed PN denotes that every clause-place \mathbf{c}_k is marked, i.e., $\mathbf{c}_k \in M^0 \forall k$, while the target marking denotes that each literal-place ℓ_i is to be marked by x_i or \bar{x}_i . That is, the inversed PN *assumes* the formula ϕ is satisfiable. This assumption is then *checked* by the “PN scan” algorithm proposed. If the truth assignment of a literal z_v in ϕ (the occurrence of a transition z_v in the “PN scan”) is incompatible, then z_v is discarded, and its negation \bar{z}_v becomes true. Because \bar{z}_v is true, z_i and z_j become false to satisfy $(\bar{z}_v \dot{\vee} z_i \dot{\vee} z_j)$, i.e., \bar{z}_i and \bar{z}_j are true. Therefore, if z_v is incompatible, a clause $(\bar{z}_v \dot{\vee} z_i \dot{\vee} z_j)$ *reduces* to the conjunction $(\bar{z}_v \wedge \bar{z}_i \wedge \bar{z}_j)$, and a 3-literal clause $(z_v \dot{\vee} z_u \dot{\vee} z_x)$ *reduces* to the 2-literal clause $(z_u \oplus z_x)$. This reduction, due to the PN approach, facilitates checking un/satisfiability.

2 Basic Definitions

This section introduces underlying tools to attack the \mathcal{P} vs \mathcal{NP} problem.

Definition 2.1. A safe PN is a tuple $PN = (P, T, F, M^0)$, in which:

- $P = \{p_1, p_2, \dots, p_m\}$ is a set of places,
- $T = \{t_1, t_2, \dots, t_n\}$ is a set of transitions such that $P \cap T = \emptyset$,
- $F \subseteq (P \times T) \cup (T \times P)$ is a flow relation,
- $M^0 \subset P$ is a set of places marked initially (the initial marking/state).
 - x^\bullet ($\bullet x$) denotes the post-set (pre-set) of $x = \{p_i, t_j\}$.

Definition 2.2. $\mathfrak{C} = \{k : |p_k^\bullet| > 1\}$ is a set of the indices of conflicts in PN , and $\mathcal{C} = \{C_k : k \in \mathfrak{C}\}$ is a family of sets of conflicts C_k , where $C_k = p_k^\bullet$.

Definition 2.3. A leveled-acyclic net [?] $\mathcal{N} = (L, P_l, T_l, F, M^0)$ is a safe PN (Defn. 2.1), in which:

- $L = \{0, 1, \dots, d\}$ is a set of levels, and $l(x)$ denotes the level of x ,
- P_l is a set of places in $l \in (L \cup \{\mathbb{S}\})$, where $P_{\mathbb{S}}$ denotes sink places,
- T_l is a set of transitions in $l \in L$,
- $M^0 = \{p \in P \mid \bullet p = \emptyset\}$, and $M_l \subseteq P_l$ denotes a set of places marked.
 - i. $\forall t \in T_l \bullet t \subseteq P_l$ (levelled \mathcal{N}) and $t^\bullet \subseteq P_{l'>l}$ (acyclic \mathcal{N}).

Fig. 1 depicts a net \mathcal{N} ; $P = \{p_1, p_2, \dots, p_{17}\}$, $T = \{t_1, t_2, \dots, t_{14}\}$, $F = \{(p_1, t_1), (p_1, t_2), \dots, (t_{14}, p_{17})\}$, and $M^0 = \{p_1, p_2, p_3, p_{11}, p_{12}, p_{13}\}$. Then, $\mathcal{C} = \{C_1, C_2, C_3\}$ denotes the sets of conflicts (conflicting transitions), where $C_1 = p_1^\bullet = \{t_1, t_2\}$, $C_2 = p_2^\bullet = \{t_3, t_4\}$, and $C_3 = p_3^\bullet = \{t_5, t_6\}$. Further, $L = \{0, 1, 2\}$, $M_0 = P_0 = \{p_1, p_2, p_3\}$, $P_1 = \{p_4, p_5, \dots, p_{13}\}$, $P_S = \{p_{17}\}$, and $T_2 = \{t_{14}\}$. Note by (i) in Defn. 2.3 that $t_8 \in T_1$ and $\bullet t_8 = \{p_5, p_{11}\} \subset P_1$, i.e., $l(t_8) = l(p_5) = l(p_{11}) = 1$ (levelled \mathcal{N}), and that $t_2 \in T_0$ and $t_2^\bullet = \{p_4, p_6\} \subset P_1$ (acyclic \mathcal{N}).

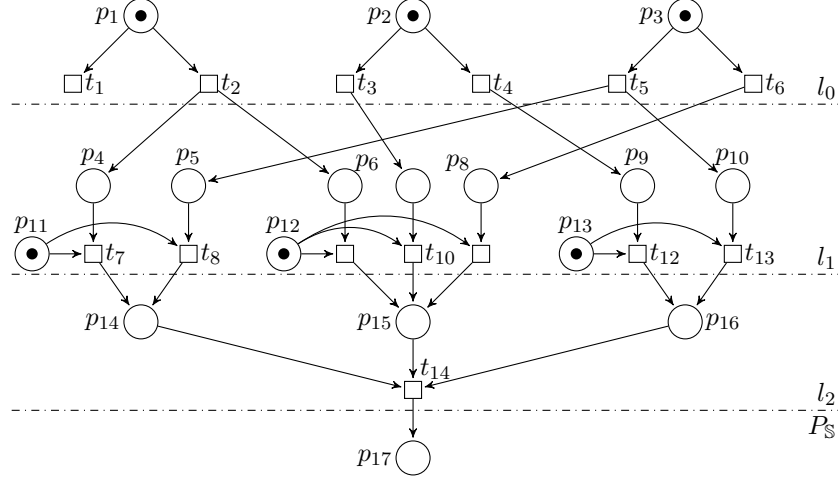


Figure 1: A (levelled-acyclic) net \mathcal{N}

Definition 2.4 (*The enabling and firing rule*). $t \in T$ is *enabled* in M if $\bullet t \subseteq M$, i.e., if each of its input places is marked in M . If t is enabled, it can *fire* (*occur*). Then, the tokens (balls in circles) are *removed* from $\bullet t$ and the new ones are *created* in t^\bullet . This firing results in a 1-step transition, and yields the consequent marking/state M' in \mathcal{N} , $M \xrightarrow{t} M'$; $M' = (M \cup t^\bullet) - \bullet t$.

Definition 2.5 (*Token game*). A safe acyclic PN is executed by a *token game*, $M^0 \xrightarrow{\sigma} M$, played from M^0 by the enabling-firing rule until no $t_j \in T$ is enabled. $M^0 \xrightarrow{\sigma} M$ denotes a k -step transition. That is, M is reached from M^0 by (firing) σ , where $\sigma = (t_{j_1}, t_{j_2}, \dots, t_{j_k})$ is a k -step transition firing sequence from M^0 . Note that no $t_j \in T$ is enabled in M , which is called a final marking, where $M = (M^0 \cup t_{j_1}^\bullet \cup t_{j_2}^\bullet \cup \dots \cup t_{j_k}^\bullet) - (\bullet t_{j_1} \cup \bullet t_{j_2} \cup \dots \cup \bullet t_{j_k})$.

For example, $T_0 = \{t_1, t_2, \dots, t_6\}$ is enabled in $M^0 = \{p_1, p_2, p_3, p_{11}, p_{12}, p_{13}\}$ in Fig. 1. If t_5 fires, then $M^0 \xrightarrow{t_5} M$, where $M = (M^0 \cup t_5^\bullet) - \bullet t_5 = \{p_1, p_2, p_5, p_{10}, p_{11}, p_{12}, p_{13}\}$. Hence, t_6 *never* fires due to C_3 . Recall that $C_1 = \{t_1, t_2\}$, $C_2 = \{t_3, t_4\}$, and $C_3 = \{t_5, t_6\}$. $M^0 \xrightarrow{\sigma} M$ is a token game by $\sigma = (t_1, t_3, t_{10}, t_5, t_8, t_{13}, t_{14})$. No $t_j \in T$ is enabled in the final marking $M = \{p_{17}\}$. $M^0 \xrightarrow{\bar{\sigma}} \bar{M}$ is another token game by $\bar{\sigma} = (t_2, t_7, t_3, t_{10}, t_6)$. No $t_j \in T$ is enabled in the final marking $\bar{M} = \{p_{14}, p_{15}, p_6, p_8, p_{13}\}$.

3 The Net \mathcal{N} of Exactly-1 3SAT: \mathcal{N}^ϕ vs \mathcal{N}^φ

Recall that reachability in safe acyclic PNs is \mathcal{NP} -complete. The proof is due to Esparza [2] based on a polynomial time construction that associates a safe acyclic PN to a boolean formula in conjunctive normal form. The net nondeterministically selects a truth assignment for the variables of the formula, and checks if the formula is true under the assignment.

Fig. 2 depicts this construction based on Esparza [2]. The net \mathcal{N} of the formula ϕ is denoted by $\mathcal{N}^\phi = (L, P_l, T_l, F, M^0)$. Note that this construction is in fact based on the reduction of Exactly-1 3SAT due to places $c_k, k = 1, 2, \dots, m$, which ensures there can exist *exactly one* true literal in the clause \mathbf{c}_k (the place \mathbf{c}_k is marked by at most one token). In \mathcal{N}^ϕ , $P_0 = \{\ell_1, \ell_2, \dots, \ell_n\}$, i.e., the (source) places in l_0 , specifies each literal ℓ_i that is either a variable x_i or its negation \bar{x}_i , i.e., $\ell_i^\bullet = \{x_i, \bar{x}_i\}$, and $T_0 = \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$ specifies the variables. Furthermore, $T_1 = \{z_{i_1 1}, z_{i_2 1}, \dots, z_{u_3 m}\}$, where $z_{v_i k} \in \{x_{v_i k}, \bar{x}_{v_i k}\}$ denotes the variable x_{v_i} or \bar{x}_{v_i} in \mathbf{c}_k , and $P_2 = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m\}$ specifies each clause $\mathbf{c}_k = (z_{v_1 k} \dot{\vee} z_{v_2 k} \dot{\vee} z_{v_3 k})$. Recall that $\dot{\vee}$ denotes *exactly one* transition $z_{v_i k}$ fires to mark \mathbf{c}_k , and that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$, where $C_i = \ell_i^\bullet$. Then, ϕ is (un)satisfiable iff M^\top is (un)reachable, i.e., iff $(M^\top \neq M^\perp) \ M^\top = M^\perp$, where $M^\top = \{\mathbf{T}^p\}$ is the target marking and M^\perp is a final marking in which no $t_j \in T$ is enabled. Note that $M^\top = M^\perp$ iff $M_\mathbb{S} = \{\mathbf{T}^p\}$ and $M_l = \emptyset \ \forall l \in L$.

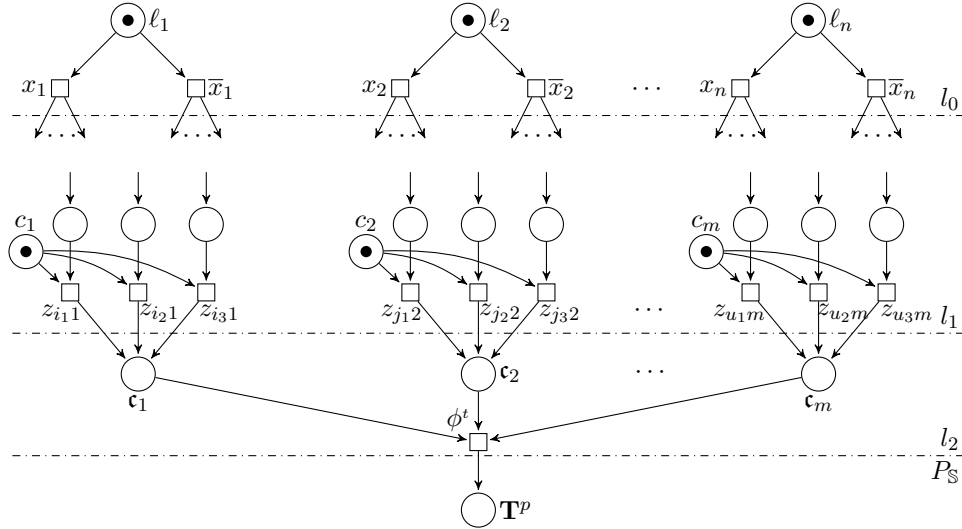


Figure 2: \mathcal{N}^ϕ ; $\phi = \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \dots \wedge \mathbf{c}_m$, $\mathbf{c}_k = z_{v_1 k} \dot{\vee} z_{v_2 k} \dot{\vee} z_{v_3 k}$, $z_{v_i k} \in \{x_{v_i k}, \bar{x}_{v_i k}\}$

On the other hand, the *inverse* of \mathcal{N}^ϕ , denoted by \mathcal{N}^φ in Fig. 3, brings about effective formulation for the \mathcal{P} vs \mathcal{NP} problem, because some 2SAT/XOR-SAT formula arisen in the \mathcal{N}^φ scan *efficiently* checks incompatibility of a truth assignment (of a transition firing). In \mathcal{N}^φ , $P_0 = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m\}$ corresponds to the *clauses* \mathbf{c}_k , and to the sets of the *conflicts* $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$.

That is, $\mathbf{c}_k^\bullet = C_k = \{z_{v_1k}, z_{v_2k}, z_{v_3k}\}$, where $z_{v_ik} \in \{x_{v_ik}, \bar{x}_{v_ik}\}$, $v_i \in \mathfrak{L}$, and $\mathfrak{L} = \{1, 2, \dots, n\}$, which is the set of the indices of the literals $\ell_i \in P_2 = \{\ell_1, \ell_2, \dots, \ell_n\}$. Further, $d = 2$ is the depth of \mathcal{N}^φ , and $|C_k| = \{2, 3\}$ because the clauses involve 2 or 3 literals in 3SAT. That is, $C_k \notin \mathcal{C}$ iff $|\mathbf{c}_k^\bullet| = 1$, and \mathbf{c}_k is called a *conjunct* rather than a clause if $|\mathbf{c}_k^\bullet| = 1$. Also, $\bullet\ell_i = \{x_i, \bar{x}_i\} \forall \ell_i \in P_2$. Note that \mathcal{N}^φ is constructed *directly* over ϕ , i.e., there is *no* need to *invert* \mathcal{N}^ϕ . Note also that \mathcal{N}^φ *assumes* that ϕ is satisfiable (due to $P_0 \subset M^0$), and that the \mathcal{N}^φ scan *checks* this assumption by the reachability of $M^\top = \{\mathbf{T}^p\}$, i.e., of $M_2 = P_2 = \{\ell_1, \ell_2, \dots, \ell_n\}$.

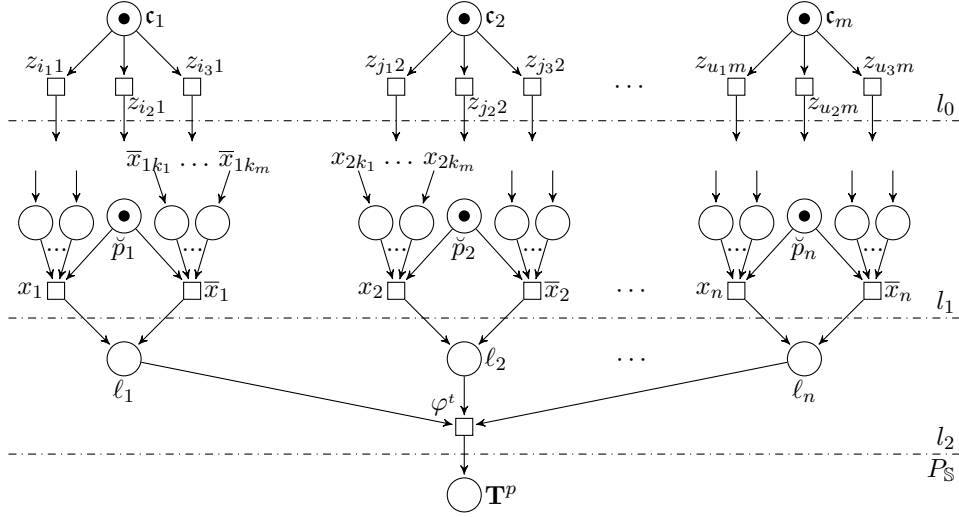


Figure 3: \mathcal{N}^φ , the inverse of \mathcal{N}^ϕ ; $\varphi = \phi = \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \dots \wedge \mathbf{c}_m - M^\top \not\equiv M^\dagger$

The next section introduces the \mathcal{N}^φ scan, which checks incompatibility of every transition firing (truth assignment of a literal) for the reachability of the marking M_2 (satisfiability of the formula ϕ). This scan is executed on the *current* \mathcal{N}^φ structure (formula ϕ). The current net \mathcal{N}^φ (formula ϕ) is obtained by discarding incompatible transitions (literals) from $\mathcal{N}^\varphi(\phi)$.

3.1 The \mathcal{N}^φ scan

Recall that a literal ℓ_i denotes a variable x_i or its negation \bar{x}_i in ϕ , and that $\ell_i \in P_2 = \{\ell_1, \dots, \ell_n\}$ is a place also in \mathcal{N}^φ . Recall also that z_{ik} is a variable (output transition) x_{ik} or \bar{x}_{ik} of the clause (place) \mathbf{c}_k . In this respect, the following definitions regard both z_i the transition and ℓ_i the place as a literal.

Definition 3.1. $z_i \in \bullet\ell_i$ and $\bar{z}_i \in \bullet\ell_i$, $\bullet\ell_i = \{x_i, \bar{x}_i\}$.

Definition 3.2. $z_i = \bar{x}_i$ iff $\bar{z}_i = x_i$, and $\bar{z}_i = \bar{x}_i$ iff $z_i = x_i$.

Definition 3.3. $\mathfrak{C}^{z_i} = \{k \in \mathfrak{C} \mid z_{ik} \Rightarrow z_i\} = \{k_1, k_2, \dots, k_m\}$ is the set of the indices of conflicts C_k involved in the z_i firing in the \mathcal{N}^φ scan (of the indices of clauses \mathbf{c}_k in the formula ϕ that the literal z_i contributes to).

Definition 3.4 (*Non/necessary z_i*). z_i is said to be necessary if it *must* fire to unmark some places \mathbf{c}_k (*must* assume true to satisfy some clauses \mathbf{c}_k).

Remark 3.1. x_i and \bar{x}_i are alternatives/nonnecessary; either x_i or \bar{x}_i can fire.

Remark 3.2 (*Truth assignment vs transition firing*). The literal z_i is assigned *true* in ϕ , $z_i = \mathbf{T}$, iff the transition z_i *fires* in the \mathcal{N}^φ scan. Then, $z_i = \mathbf{F}$ iff z_i does not fire, denoted by $\neg z_i$, i.e., $z_i = \mathbf{F}$ iff $\neg z_i$. Therefore, $\bar{z}_i = \mathbf{F}$ iff $\neg \bar{z}_i$.

Lemma 3.1 (Necessary/Conjunct z_i). z_i is necessary iff $\mathbf{c}_k^\bullet = \{z_{ik}\}$.

Definition 3.5. $\phi(z_i) = \phi \wedge z_i$, where $z_i \in \bullet \ell_i$ is a literal in the formula ϕ .

Lemma 3.2 (Reduction of clauses). $\phi(z_i)$ entails $(\bar{z}_i \dot{\vee} z_v \dot{\vee} z_y) \rightarrow (z_v \oplus z_y)$ and $(z_i \dot{\vee} z_j \dot{\vee} \bar{z}_u) \rightarrow (z_i \wedge \bar{z}_j \wedge z_u)$.

Proof: Because $z_i = \mathbf{T}$ to satisfy $\phi(z_i) = \phi \wedge z_i$, it is the case that $\bar{z}_i = \mathbf{F}$. Further, because $z_i = \mathbf{T}$, it is the case that $z_j = \bar{z}_u = \mathbf{F}$ to satisfy $(z_i \dot{\vee} z_j \dot{\vee} \bar{z}_u)$ due to exactly-1 disjunction $\dot{\vee}$. Therefore, $\bar{z}_j = z_u = \mathbf{T}$. Consequently, $\phi(z_i)$ entails that every clause $(z_i \dot{\vee} z_j \dot{\vee} \bar{z}_u)$ *reduce* to the conjunction $(z_i \wedge \bar{z}_j \wedge z_u)$, i.e., $(z_i \dot{\vee} z_j \dot{\vee} \bar{z}_u) \rightarrow (z_i \wedge \bar{z}_j \wedge z_u)$, and that every 3-literal clause $(\bar{z}_i \dot{\vee} z_v \dot{\vee} z_y)$ *reduce* to the 2-literal clause $(z_v \oplus z_y)$, i.e., $(\bar{z}_i \dot{\vee} z_v \dot{\vee} z_y) \rightarrow (z_v \oplus z_y)$. Note that $(z_i \dot{\vee} z_j \dot{\vee} \bar{z}_u) \not\equiv (z_i \oplus z_j \oplus \bar{z}_u)$, and that $(z_v \dot{\vee} z_y) \equiv (z_v \oplus z_y)$. \square

Definition 3.6 (*General vs special nets*). $\{x_{ik}, \bar{x}_{ik}\} \not\subseteq C_k \forall k \in \mathfrak{C} \forall i \in \mathfrak{L}$ in a general \mathcal{N}^φ (structure), while $\{x_{ik}, \bar{x}_{ik}\} \subseteq C_k$ for some k in a special \mathcal{N}^φ .

Lemma 3.3 (Conversion of a special formula/net). $\phi(\bar{z}_j)$ converts a special formula ϕ to the general formula $\phi \equiv \phi(\bar{z}_j)$, where $C_{k \in \mathfrak{C}} = \{z_{jk}, x_{ik}, \bar{x}_{ik}\}$.

Proof: If z_j marks ℓ_j , then *neither* x_i *nor* \bar{x}_i marks ℓ_i , i.e., $\ell_i \notin M_2$ in \mathcal{N}^φ . Therefore, z_j cannot fire, and to be discarded. Because z_j is discarded (false), \bar{z}_j becomes necessary (true). Because \bar{z}_j is necessary, $\mathbf{c}_k^\bullet = \{\bar{z}_{jk}\}$ for some k (Lemma 3.1), which is the case in the formula $\phi(\bar{z}_j) = \phi \wedge \bar{z}_j$ (Defn. 3.5). Consequently, a special ϕ is converted to the general ϕ by means of $\phi(\bar{z}_j)$. \square

For example, $\phi = \varphi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge x_4$ is a general formula, while $\phi' = \varphi' = (x_1 \dot{\vee} \bar{x}_3 \dot{\vee} x_4) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_2) \wedge (x_2 \dot{\vee} \bar{x}_3)$ is a special formula due to $C_2 = \mathbf{c}_2^\bullet = \{x_{12}, \bar{x}_{22}, x_{22}\}$. Note that, in \mathcal{N}^φ , $\mathfrak{C}^{x_1} = \{1, 2\}$ and $\mathfrak{C}^{x_4} = \emptyset$ (Defn. 3.3) due to $\mathbf{c}_4^\bullet = \{x_{44}\}$ (Lemma 3.1), while $\mathfrak{C}^{x_4} = \{1\}$ in $\mathcal{N}^{\varphi'}$. Then, *neither* \bar{x}_2 *nor* x_2 marks ℓ_2 , i.e., ℓ_2 is *not* marked ($\ell_2 \notin M_2$), if x_1 marks ℓ_1 in $\mathcal{N}^{\varphi'}$. Therefore, ϕ' is converted to the general formula (Lemma 3.3) by discarding x_1 , i.e., by means of $\phi'(\bar{x}_1)$. Consequently, $\phi' \equiv \phi'(\bar{x}_1) = \phi' \wedge \bar{x}_1 \equiv (\bar{x}_3 \oplus x_4) \wedge (\bar{x}_2 \oplus x_2) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge \bar{x}_1 \equiv (\bar{x}_3 \oplus x_4) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge \bar{x}_1$.

Recall that some 2SAT/XOR-SAT formula arises in the \mathcal{N}^φ scan to check incompatibility of a firing. For example, consider incompatibility of \bar{x}_1 for the formula ϕ above. If \bar{x}_1 fires, then $\neg x_1$, i.e., \bar{x}_{31} and $(\bar{x}_{22} \text{ xor } x_{32})$ unmark the places \mathbf{c}_1 and \mathbf{c}_2 . Thus, $\bar{x}_1 \Rightarrow \phi(\bar{x}_1)$, where $\phi(\bar{x}_1) = \bar{x}_1 \wedge \bar{x}_3 \wedge (\bar{x}_2 \oplus x_3) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge x_4$. Then, \bar{x}_1 is *incompatible* if $\phi(\bar{x}_1)$ is *unsatisfiable*. Theorems in the sequel incorporate this feature, which arises due to the PN approach.

Definition 3.7. $\varphi(z_v)$ denotes the *partial* effect of the z_v firing.

Definition 3.8. $\varphi(\neg z_v)$ denotes the effect of the z_v unfiring.

Definition 3.9. Let $\mathfrak{C}^{z_v} = \{k_1, k_2, \dots, k_r\}$ for $\varphi(z_v)$ and $\varphi(\neg z_v)$, and let $\mathbf{c}_{k_1} = (z_v \dot{\vee} z_{i_1} \dot{\vee} z_{i_2})$, $\mathbf{c}_{k_2} = (z_v \dot{\vee} z_{j_1} \dot{\vee} z_{j_2}) \cdots \mathbf{c}_{k_r} = (z_v \dot{\vee} z_{u_1} \dot{\vee} z_{u_2})$.

Lemma 3.4. $z_v \Rightarrow \varphi(z_v) = \bar{z}_{i_1} \wedge \bar{z}_{i_2} \wedge \bar{z}_{j_1} \wedge \bar{z}_{j_2} \wedge \cdots \wedge \bar{z}_{u_1} \wedge \bar{z}_{u_2}$.

Proof: $z_v \Rightarrow (z_v \wedge \bar{z}_{i_1} \wedge \bar{z}_{i_2}) \wedge (z_v \wedge \bar{z}_{j_1} \wedge \bar{z}_{j_2}) \wedge \cdots \wedge (z_v \wedge \bar{z}_{u_1} \wedge \bar{z}_{u_2})$ due to the reduction of clauses (Lemma 3.2). Therefore, $z_v \Rightarrow \varphi(z_v)$. \square

Lemma 3.5. $\neg z_v \Rightarrow \varphi(\neg z_v) = (z_{i_1} \oplus z_{i_2}) \wedge (z_{j_1} \oplus z_{j_2}) \wedge \cdots \wedge (z_{u_1} \oplus z_{u_2})$.

Proof: If $\neg z_v$, then $(z_{i_1} \oplus z_{i_2}) \wedge (z_{j_1} \oplus z_{j_2}) \wedge \cdots \wedge (z_{u_1} \oplus z_{u_2})$ must fire. \square

Lemma 3.6 (The overall effect of the z_v firing). $\psi(z_v) = z_v \wedge \varphi(z_v) \wedge \varphi(\neg \bar{z}_v)$.

Proof: z_v iff $\neg \bar{z}_v$ (z_v fires iff \bar{z}_v does not fire). Then, $z_v \Rightarrow \neg \bar{z}_v$. Also, $z_v \Rightarrow z_v$, $z_v \Rightarrow \varphi(z_v)$, and $\neg \bar{z}_v \Rightarrow \varphi(\neg \bar{z}_v)$. Therefore, $z_v \Rightarrow z_v \wedge \varphi(z_v) \wedge \varphi(\neg \bar{z}_v)$. \square

Remark 3.3. $\mathfrak{C}^{z_v} \cap \mathfrak{C}^{\bar{z}_v} = \emptyset$ because $\{x_{vk}, \bar{x}_{vk}\} \not\subseteq C_k$ due to Lemma 3.3.

Definition 3.10. z_v is incompatible to assume true iff $\phi(z_v) \stackrel{\text{def}}{=} z_v \wedge \phi = \perp$.

Lemma 3.7. $\phi(z_v) \equiv \psi(z_v) \wedge \phi'$, where ϕ' is a sub-formula of ϕ .

Lemma 3.8 (A sufficient condition for incompatibility of z_v). If $\psi(z_v) = \perp$, then $\phi(z_v) = \perp$, i.e., z_v is incompatible.

Lemma 3.9 (The scope $\Psi(z_v)$ over ϕ). $\phi(z_v) \equiv \Psi(z_v) \wedge \phi^*$.

Proof: $\phi(z_v)$ entails reduction of clauses to conjunctions (Lemma 3.2). Each conjunction entails further reductions, i.e., $z_v \Rightarrow z_v \wedge z_{v_1} \wedge \cdots \wedge z_{v_n}$. Therefore, $\Psi(z_v) = \psi(z_v) \wedge \psi(z_{v_1}) \wedge \cdots \wedge \psi(z_{v_n})$ and $\phi(z_v) \equiv \Psi(z_v) \wedge \phi^*$ (Lemma 3.7). \square

Remark 3.4. Lemma 3.9 generalizes Lemma 3.7. ϕ^* is a sub-formula of ϕ , and can be empty. If ϕ^* is empty, the scope $\Psi(z_v)$ is said to cover the formula ϕ , i.e., $\phi(z_v) \equiv \Psi(z_v)$. Otherwise, ϕ^* is said to be beyond the scope $\Psi(z_v)$.

For example, consider $\phi(x_1)$ over $\phi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$. Then, $\varphi(x_1) = (x_3) \wedge (x_2 \wedge \bar{x}_3)$ and $\varphi(\neg x_1) = (\bar{x}_3) \wedge (\bar{x}_2 \oplus x_3)$, while $\varphi(\bar{x}_1)$ and $\varphi(\neg \bar{x}_1)$ are empty as $\mathfrak{C}^{\bar{x}_1} = \emptyset$. Thus, $\phi(x_1) = \phi \wedge x_1$ entails $x_1 \Rightarrow \psi(x_1)$, in which $\psi(x_1) = x_1 \wedge \varphi(x_1) \wedge \varphi(\neg \bar{x}_1) = x_1 \wedge x_3 \wedge x_2 \wedge \bar{x}_3$ (Lemma 3.6). Further, $\phi(x_1) \equiv \psi(x_1) \wedge \phi'$ (Lemma 3.7), where $\phi' = \mathbf{c}_5 = (x_2 \dot{\vee} \bar{x}_3)$. Therefore, x_1 is incompatible as $\psi(x_1) = \perp$ (Lemma 3.8) due to $x_3 \wedge \bar{x}_3$, i.e., if $x_1 = \mathbf{T}$, then ϕ is unsatisfiable (M_2 is unreachable as $\ell_3 \notin M_2$). Similarly, $\phi(\bar{x}_1)$ entails $\bar{x}_1 \Rightarrow \psi(\bar{x}_1)$, in which $\psi(\bar{x}_1) = \bar{x}_1 \wedge \varphi(\bar{x}_1) \wedge \varphi(\neg x_1) = \bar{x}_1 \wedge \bar{x}_3 \wedge (\bar{x}_2 \oplus x_3)$. Then, $\phi(\bar{x}_1) \equiv \psi(\bar{x}_1) \wedge \phi' = \bar{x}_1 \wedge \bar{x}_3 \wedge (\bar{x}_2 \oplus x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$, where $\phi' = \mathbf{c}_4 = (x_2 \dot{\vee} \bar{x}_3)$. Since \bar{x}_3 is a conjunct for $\phi(\bar{x}_1)$, i.e., $\mathbf{c}_5^* = \{\bar{x}_{32}\}$, it is the case that $\bar{x}_3 \Rightarrow \psi(\bar{x}_3)$, in which $\psi(\bar{x}_3) = \bar{x}_3 \wedge \varphi(\bar{x}_3) \wedge \varphi(\neg x_3) = \bar{x}_3 \wedge \bar{x}_2$, where $\varphi(\bar{x}_3) = (\bar{x}_2)$ by $\mathfrak{C}^{\bar{x}_3} = \{4\}$ and $\varphi(\neg x_3) = (\bar{x}_2)$ by $\mathfrak{C}^{x_3} = \{3\}$ in $\phi(\bar{x}_1)$. Therefore, the scope of \bar{x}_1 over ϕ is $\Psi(\bar{x}_1) = \psi(\bar{x}_1) \wedge \psi(\bar{x}_3) = \bar{x}_1 \wedge \bar{x}_3 \wedge \bar{x}_2$, and $\phi(\bar{x}_1) \equiv \Psi(\bar{x}_1) \wedge \phi^*$, where ϕ^* is empty. Consequently, $\Psi(\bar{x}_1)$ covers ϕ , i.e., $\phi(\bar{x}_1) \equiv \Psi(\bar{x}_1) \equiv \phi$.

Definition 3.11. X_s denotes the *current* net/formula for the s^{th} \mathcal{N}^φ scan.

Theorem 3.10. z_v becomes incompatible for ϕ_s iff $\Psi_s(z_v) = \perp$.

Proof: The validity of the theorem follows from $\phi_s(z_v) = z_v \wedge \phi_s = \perp$ (Defn. 3.10) and from $\phi_s(z_v) \equiv \Psi_s(z_v) \wedge \phi_s^*$ (Lemma 3.8/3.9), which depends on the soundness of the $\Psi_s(z_v)$ construction over $\phi_s(z_v)$, on the satisfiability of $\Psi_s(z_v)$ and ϕ_s^* , and on the monotonicity of $\Psi_s(z_v)$. Related to the soundness, the $\Psi_s(z_v)$ construction is a *deterministic* chain of reductions of some clauses to conjunctions, and of some 3-literal clauses to 2-literal clauses, specified in the algorithm **Incompatible** (z_v) below. Related to the satisfiability of $\Psi_s(z_v)$ and ϕ_s^* , if $\Psi_s(z_v) = \mathbf{T}$ and $\phi_s^* = \perp$, then not only z_v is incompatible, but also $\phi_s = \perp$, i.e., ϕ is unsatisfiable. Because unsatisfiability of ϕ_s^* is already checked by the algorithm **Scan**, introduced in the sequel, it is *irrelevant* to check if $\phi_s^* = \perp$ for the incompatibility of z_v . Related to the monotonicity of $\Psi_s(z_v)$, because $\Psi_s(z_v)$ is the formula arisen from $\Psi_{\hat{s}}(z_v)$ by *reducing* some 3/2-literal clauses to 2/1-literal clauses, if $\Psi_{\hat{s}}(z_v) = \perp$, then $\Psi_s(z_v) = \perp$ for all $s > \hat{s}$, i.e., incompatible z_v *never* becomes *compatible* again. \square

Corollary 3.11. $\phi_s(z_v) \equiv \phi$ is satisfiable if $\Psi_s(z_v) = \mathbf{T}$ and ϕ_s^* is empty.

Algorithm 1 **Incompatible** (z_v) ▷ Construction of $\Psi_s(z_v)$ over $\phi_s(z_v)$

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1:  $E \leftarrow \{z_v\}$    ▷  $E$  is a set of conjuncts emerged during the reductions in  $\phi_s(z_v)$ 
2: while  ${}^c\phi_s^3(z_v)$  is not empty and  $E \neq E'$  do   ▷ Reductions start off
3:   for all  $z_j \in (E - E')$  do   ▷  $E'$  denotes conjuncts already reduced clauses
4:     for all  $k \in \mathfrak{C}^{z_j}$  do   ▷  $z_j = \mathbf{T}$ ; Determine  $\varphi_s(z_j)$  (Lemma 3.4)
5:       for all  $z_{ik} \in (C_k - \{z_{jk}\})$  do   ▷ Reduce  $\mathbf{c}_k$  to conjunction
6:          $E_k \leftarrow (E_k \cup \{\bar{z}_i\})$    ▷  $E_k$  is a set of conjuncts reduced from  $\mathbf{c}_k$ 
7:       end for
8:        $E \leftarrow (E \cup E_k)$ ,  $\mathbf{c}_k^* \leftarrow \emptyset$    ▷  $\phi_s(z_v)$  is updated (16);  $\mathbf{c}_k^* = \emptyset$ 
9:     end for   ▷  $(z_j \dot{\vee} z_{i_1} \dot{\vee} z_{i_2}) \rightarrow (z_j \wedge \bar{z}_{i_1} \wedge \bar{z}_{i_2})$  (Lemma 3.2)
10:    if  $\{x_i, \bar{x}_i\} \subseteq E$  then return  $z_v$  is incompatible   ▷  $x_i \wedge \bar{x}_i = \perp$ 
11:    for all  $k \in \mathfrak{C}^{\bar{z}_j}$  do   ▷  $\bar{z}_j = \mathbf{F}$ ; Determine  $\varphi_s(\neg \bar{z}_j)$  (Lemma 3.5)
12:       $\mathbf{c}_k^* \leftarrow (\mathbf{c}_k^* - \{\bar{z}_{jk}\})$    ▷ Reduce 3/2-literal  $\mathbf{c}_k$  to 2/1-literal  $\mathbf{c}_k$ 
13:      if  $\mathbf{c}_k^* = \{z_{uk}\}$  then  $E \leftarrow (E \cup \{z_u\})$ ,  $\mathbf{c}_k^* \leftarrow \emptyset$ 
14:    end for   ▷  $(\bar{z}_j \dot{\vee} z_{u_1} \dot{\vee} z_{u_2}) \rightarrow (z_{u_1} \oplus z_{u_2})$  (Lemma 3.2)
15:     $E' \leftarrow (E' \cup \{z_j\})$    ▷  $z_j$ , and  $\bar{z}_j$ , has reduced clauses;  $E'$  is updated
16:    Determine the current  ${}^c\phi_s(z_v) \equiv \phi_s(z_v)$ , and Update  $\mathfrak{C}$  for  ${}^c\phi_s(z_v)$ 
17:  end for
18: end while   ▷ Reductions in  $\phi_s(z_v)$  terminate;  ${}^c\phi_s^3(z_v)$  is empty or  $E = E'$ 
19: if  $\Psi_s(z_v) = (\bigwedge_{z_u \in E}) \wedge {}^c\phi_s^2(z_v) = \perp$  then return  $z_v$  is incompatible
20: else return  $z_v$  is not yet incompatible   ▷ It can be incompatible for  $\phi_{\hat{s} > s}$ 
21: end if   ▷ Because reductions in  $\phi_s(z_v)$  terminate,  $\Psi_s(z_v)$  can be determined

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In the algorithm **Incompatible** (z_v), $\Psi_s(z_v) = (\bigwedge_{z_u \in E}) \wedge {}^c\phi_s^2(z_v)$ becomes the scope over $\phi_s(z_v)$ (L:19), when the reductions started off (L:2) terminate (L:18), i.e., if the formula of 3-literal clauses over $\phi_s(z_v)$, ${}^c\phi_s^3(z_v)$, is empty or $E = E'$. $\Psi_s(z_v)$ incorporates two formulas based on $\varphi_s(z_j)$ and $\varphi_s(\neg \bar{z}_j)$ due to every conjunct z_j *emerged during* the reductions. The former, $\bigwedge_{z_u \in E} = z_{u_1} \wedge \varphi_s(z_{u_1}) \wedge \cdots \wedge z_{u_n} \wedge \varphi_s(z_{u_n}) \wedge R$, is the conjunction due to $\varphi_s(z_j)$ (L:4-9) and R (L:13), while the latter, ${}^c\phi_s^2(z_v)$, is the formula of 2-literal clauses due to $\varphi_s(\neg \bar{z}_j)$ (L:11-14), as well as due to the 2-literal clauses *not* reduced, i.e., *remained* in $\phi_s(z_v)$. Consequently, $\phi_s(z_v) \equiv \Psi_s(z_v) \wedge {}^c\phi_s^3(z_v)$, where ${}^c\phi_s^3(z_v)$ is beyond the scope $\Psi_s(z_v)$. If $\Psi_s(z_v) = \perp$, a 2SAT/XOR-SAT formula, then z_v *becomes incompatible* for ϕ_s (L:19), else z_v is *not yet* incompatible for ϕ_s .

The \mathcal{N}^φ scan, the algorithm **Scan** \mathcal{N}_s^φ , is introduced below. Recall that $\{x_{ik}, \bar{x}_{ik}\} \not\subseteq C_k$ by the conversion of a special net to a general net (Lemma 3.3). Note that there is no need to check incompatibility of \bar{z}_v if $\bullet\ell_v = \{\bar{z}_v\}$, i.e., $|\bullet\ell_v| = 1$ (L:4 in **Scan** \mathcal{N}_s^φ), because its incompatibility has already been checked in the algorithm **Discard** (z_v). More precisely, if z_v is incompatible, then it is discarded (L:6 in **Scan** \mathcal{N}_s^φ), i.e., the \bar{z}_v firing initiates a chain of reductions (L:2-12 in **Discard** (z_v)). Then, \bar{z}_v is incompatible, i.e., $\phi = \perp$ (L:8 in **Discard** (z_v)) or \bar{z}_v is not incompatible, i.e., \mathcal{N}^φ is updated (L:14).

Algorithm 2 **Scan** \mathcal{N}_s^φ \triangleright Incompatibility of all $z_v \in \bullet\ell_v \ \forall v \in \mathfrak{L}$ are checked

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1: for all  $\mathbf{c}_k = \{\bar{z}_{vk}\}$  &  $|\bullet\ell_v| = 2$  do  $\triangleright \bar{z}_v$  is necessary (Lemma 3.1), i.e.,  $\bar{z}_v = \mathbf{T}$ 
2:   Discard ( $z_v$ )       $\triangleright z_v$  is incompatible as  $\bar{z}_v$  is necessary, i.e.,  $z_v = \mathbf{F}$ 
3: end for       $\triangleright$  If  $|\bullet\ell_v| = 1$ , then  $z_v$  has already been discarded
4: for all  $\ell_v \in P_2$  &  $|\bullet\ell_v| = 2$  do
5:   for all  $z_v \in \bullet\ell_v = \{x_v, \bar{x}_v\}$  do
6:     if Incompatible ( $z_v$ ) then Discard ( $z_v$ )       $\triangleright$  Theorem 3.10
7:   end for
8: end for       $\triangleright$  If  $|\bullet\ell_v| = 1$ , then incompatibility of  $\bar{z}_v$  has already been checked
9: return  $\{\bullet\ell_1, \bullet\ell_2, \dots, \bullet\ell_n\}$  satisfies  $\phi$ 

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The algorithm **Discard** (z_v) is introduced below, in which the \bar{z}_v firing initiates the same chain of reductions as the one in **Incompatible** (\bar{z}_v). That is, if z_v is *incompatible*, then it is false and *discarded*, i.e., \bar{z}_v is a *conjunct*. Therefore, $\bar{z}_v = \mathbf{T}$ and $z_{i_1} = z_{i_2} = \mathbf{F}$, i.e., $\bar{z}_{i_1} = \bar{z}_{i_2} = \mathbf{T}$, to satisfy each clause $(\bar{z}_v \dot{\vee} z_{i_1} \dot{\vee} z_{i_2})$. As a result, each clause $\mathbf{c}_k = (\bar{z}_v \dot{\vee} z_{i_1} \dot{\vee} z_{i_2})$ reduces to the conjunction $(\bar{z}_v \wedge \bar{z}_{i_1} \wedge \bar{z}_{i_2})$ (L:2-7). Then, $N_k = \{\bar{z}_{i_1}, \bar{z}_{i_2}\}$ is a set of conjuncts over $(\bar{z}_{i_1} \wedge \bar{z}_{i_2})$ reduced from \mathbf{c}_k (L:3-5), while N is the set of all conjuncts over the formula ϕ (L:1). If $\{x_i, \bar{x}_i\} \subseteq N$, i.e., if $x_i \wedge \bar{x}_i = \perp$, then $\phi_s \equiv \phi$ is unsatisfiable (L:8). Otherwise, z_{vk} is discarded from each clause \mathbf{c}_k (L:9-12), and ϕ/\mathcal{N}^φ is updated (L:14). Note that some clauses reduce to conjunctions iff N is updated (L:6/11). In this case, Lines (1-3) are executed in **Scan** \mathcal{N}_s^φ , where $|\bullet\ell_v| = 1$ indicates $z_v \in N$ has already been discarded.

Algorithm 3 Discard (z_v)	\triangleright Incompatible/nonnecessary z_v from $\mathcal{N}_s^\varphi/\phi_s$
1: $N \leftarrow (N \cup \{\bar{z}_v\})$	$\triangleright \bar{z}_v$ is necessary/conjunct, N is their set over \mathcal{N}^φ/ϕ
2: for all $k \in \mathfrak{C}^{\bar{z}_v}$ do	$\triangleright \bar{z}_v$ is necessary; Determine $\varphi_s(\bar{z}_v)$ (Lemma 3.4)
3: for all $z_{ik} \in (C_k - \{\bar{z}_{vk}\})$ do	\triangleright Reduce \mathbf{c}_k to conjunction due to \bar{z}_v
4: $N_k \leftarrow (N_k \cup \{\bar{z}_i\})$	$\triangleright N_k$ is a set of conjuncts \bar{z}_i reduced from \mathbf{c}_k
5: end for	
6: $N \leftarrow (N \cup N_k), \mathbf{c}_k^\bullet \leftarrow \emptyset$	$\triangleright \mathbf{c}_k$ is reduced/ ϕ_s is updated; $\mathbf{c}_k^\bullet = \emptyset$
7: end for	$\triangleright (\bar{z}_v \dot{\vee} z_{i_1} \dot{\vee} z_{i_2}) \rightarrow (\bar{z}_v \wedge \bar{z}_{i_1} \wedge \bar{z}_{i_2})$ (Lemma 3.2)
8: if $\{x_i, \bar{x}_i\} \subseteq N$ then return ϕ is unsatisfiable	$\triangleright \phi_s \equiv \phi = \perp$
9: for all $k \in \mathfrak{C}^{z_v}$ do	$\triangleright z_v$ is incompatible; Determine $\varphi_s(\neg z_v)$ (Lemma 3.5)
10: $\mathbf{c}_k^\bullet \leftarrow (\mathbf{c}_k^\bullet - \{z_{vk}\})$	\triangleright Reduce 3/2-literal \mathbf{c}_k to 2/1-literal \mathbf{c}_k
11: if $\mathbf{c}_k^\bullet = \{z_{uk}\}$ then $N \leftarrow (N \cup \{z_u\}), \mathbf{c}_k^\bullet \leftarrow \emptyset$	
12: end for	$\triangleright (z_v \dot{\vee} z_{u_1} \dot{\vee} z_{u_2}) \rightarrow (z_{u_1} \oplus z_{u_2})$ (Lemma 3.2)
13: $\bullet \ell_v \leftarrow \{\bar{z}_v\}$	\triangleright Discard z_v , i.e., $\bullet \ell_v = \{\bar{z}_v\}$
14: Determine ϕ_{s+1} and $\mathcal{N}_{s+1}^\varphi$	$\triangleright N$ is updated or 3-literal \mathbf{c}_k is reduced
15: Scan $\mathcal{N}_{s+1}^\varphi$	\triangleright Re-scan \mathcal{N}^φ due to \mathcal{N}^φ re-structured

Let $\phi = \varphi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$, and check if $\phi(x_1) = \perp$, $\phi(x_1) = \phi \wedge x_1$, i.e., consider **Incompatible** (x_1). Then, the x_1 firing initiates a chain of reductions in $\mathcal{N}_1^\varphi = \mathcal{N}^\varphi$. That is, $x_1 \Rightarrow \psi(x_1)$, where $\psi(x_1) = x_1 \wedge \varphi(x_1) \wedge \varphi(\neg \bar{x}_1)$ (Lemma 3.6). Therefore, $E \leftarrow \{x_1\}$ (L:1), $\mathfrak{C}^{x_1} = \{1, 2\}$ (L:4), i.e., C_1 and C_2 participate in the x_1 firing (x_1 contributes to the clauses \mathbf{c}_1 and \mathbf{c}_2), $E_1 \leftarrow \{x_3\}$ (L:6), $E \leftarrow \{x_1, x_3\}$, $\mathbf{c}_1^\bullet \leftarrow \emptyset$ (L:8), and $E_2 \leftarrow \{x_2, \bar{x}_3\}$ (L:6), $E \leftarrow \{x_1, x_3, x_2, \bar{x}_3\}$, $\mathbf{c}_2^\bullet \leftarrow \emptyset$ (L:8). Because $\{x_3, \bar{x}_3\} \subset E$ (L:10), i.e., $x_3 \wedge \bar{x}_3 = \perp$, $\phi(x_1) = \perp$ (x_1 becomes incompatible for ϕ). Note that $\varphi(\neg \bar{x}_1)$ is empty. Consequently, $\bigwedge_{z_u \in E} = x_1 \wedge \varphi(x_1)$, where $\varphi(x_1) = x_3 \wedge x_2 \wedge \bar{x}_3$ (Lemma 3.4) is the formula over $E_1 \cup E_2$, i.e., $x_1 \Rightarrow (x_1 \wedge x_3) \wedge (x_1 \wedge x_2 \wedge \bar{x}_3)$. That is, $(x_1 \dot{\vee} \bar{x}_3) \rightarrow (x_1 \wedge x_3)$, and $(x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \rightarrow (x_1 \wedge x_2 \wedge \bar{x}_3)$.

Because x_1 is *incompatible*, it is *discarded* (\bar{x}_1 becomes a *conjunct*). Then, \bar{x}_1 initiates a chain of reductions. That is, $\bar{x}_1 \Rightarrow \psi(\bar{x}_1)$, in which $\psi(\bar{x}_1) = \bar{x}_1 \wedge \varphi(\bar{x}_1) \wedge \varphi(\neg x_1) = \bar{x}_1 \wedge (\bar{x}_3) \wedge (\bar{x}_2 \oplus x_3)$. Therefore, $\phi_2 = \psi(\bar{x}_1) \wedge (x_2 \dot{\vee} \bar{x}_3)$, in which $\mathbf{c}_1^\bullet = \{\bar{x}_{11}\}$, $\mathbf{c}_2^\bullet = \{\bar{x}_{32}\}$, i.e., \bar{x}_3 is a *conjunct*, $\mathbf{c}_3^\bullet = \{\bar{x}_{23}, x_{33}\}$, and $\mathbf{c}_4^\bullet = \{x_{24}, \bar{x}_{34}\}$. Subsequently, $\bar{x}_3 \Rightarrow \psi_2(\bar{x}_3)$, in which $\psi_2(\bar{x}_3) = \bar{x}_3 \wedge \varphi_2(\bar{x}_3) \wedge \varphi_2(\neg x_3)$, where $\varphi_2(\bar{x}_3) = \bar{x}_2$ by $\mathfrak{C}^{\bar{x}_3} = \{4\}$ and $\varphi_2(\neg x_3) = \bar{x}_2$ by $\mathfrak{C}^{x_3} = \{3\}$. Therefore, $\phi_3 = \psi(\bar{x}_1) \wedge \psi_2(\bar{x}_3) = \bar{x}_1 \wedge \bar{x}_3 \wedge \bar{x}_2$. Note that $\phi_3 \equiv \phi$. Note also that $\phi \rightarrow \phi_2$ if $(x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \rightarrow (\bar{x}_2 \oplus x_3)$, and $\phi_2 \rightarrow \phi_3$ if $(x_2 \dot{\vee} \bar{x}_3) \rightarrow (\bar{x}_2 \wedge \bar{x}_3)$.

3.2 An Illustrative Example

Fig. 4b depicts \mathcal{N}^φ for $\phi = \varphi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$. In \mathcal{N}^φ , $P_0 = M_0 = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$, $P_1 = \{p_{11}, \dots, \bar{p}_{33}\} \cup \{\check{p}_1, \check{p}_2, \check{p}_3\}$, $P_2 = \{\ell_1, \ell_2, \ell_3\}$, $M^0 = P_0 \cup \{\check{p}_1, \check{p}_2, \check{p}_3\}$, and $C_k = \mathbf{c}_k^\bullet \forall k \in \mathfrak{C} = \{1, 2, 3\}$, e.g., $C_1 = \{x_{11}, \bar{x}_{31}\}$.

Scan \mathcal{N}^φ : Because $|\mathbf{c}_k^\bullet| \neq 1 \ \forall k$ (L:1-3), i.e., all $z_i \in \bullet \ell_i \ \forall i$ are nonnecessary, Incompatible $\{x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3\}$ are executed (L:4-8). Recall that \mathcal{N}_s^φ is the net for the s^{th} scan, $\mathcal{N}^\varphi = \mathcal{N}_1^\varphi$, and that the order of incompatibility check is insignificant due to the monotonicity of $\Psi_s(z_v)$ (Theorem 3.10).

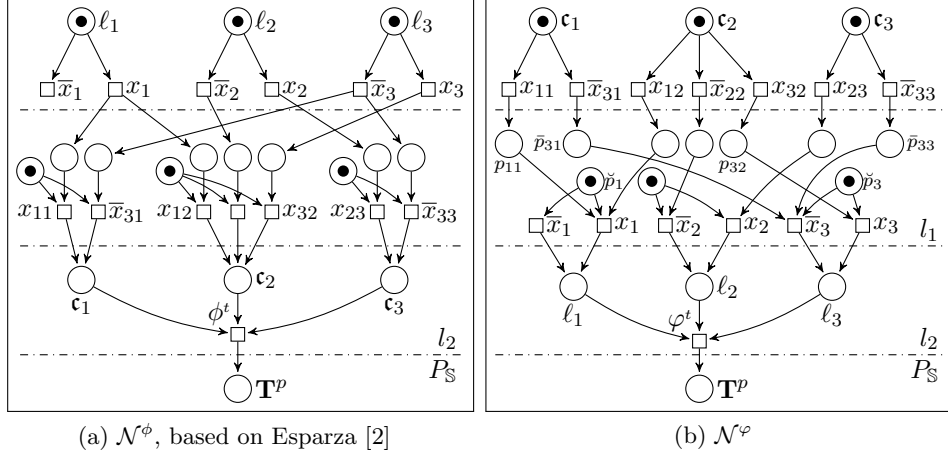


Figure 4: \mathcal{N}^ϕ and \mathcal{N}^φ ; $\phi = \varphi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$

Incompatible (\bar{x}_2) in Scan \mathcal{N}^φ ; $\phi(\bar{x}_2) = \phi \wedge \bar{x}_2$: $E \leftarrow \{\bar{x}_2\}$ (L:1), $\mathfrak{C}^{\bar{x}_2} = \{2\}$ (L:4), $E_2 \leftarrow \{\bar{x}_1, \bar{x}_3\}$ (L:6), and $E \leftarrow \{\bar{x}_2, \bar{x}_1, \bar{x}_3\}$, $\mathbf{c}_2^\bullet \leftarrow \emptyset$ (L:8). $\mathfrak{C}^{x_2} = \{3\}$ (L:11), $\mathbf{c}_3^\bullet \leftarrow (\{x_{23}, \bar{x}_{33}\} - \{x_{23}\})$ (L:12). Because $\mathbf{c}_3^\bullet = \{\bar{x}_{33}\}$, $E \leftarrow (E \cup \{\bar{x}_3\})$ and $\mathbf{c}_3^\bullet \leftarrow \emptyset$ (L:13). Then, ${}^c\phi(\bar{x}_2) = \bar{x}_2 \wedge \bar{x}_1 \wedge \bar{x}_3 \wedge (x_1 \dot{\vee} \bar{x}_3)$ (L:16), where $\mathbf{c}_1^\bullet = \{\bar{x}_{21}\}$, $\mathbf{c}_2^\bullet = \{\bar{x}_{12}\}$, $\mathbf{c}_3^\bullet = \{\bar{x}_{33}\}$, $\mathbf{c}_4^\bullet = \{x_{14}, \bar{x}_{34}\}$, and $\mathfrak{C} = \{4\}$. Because ${}^c\phi^3(\bar{x}_2)$ is empty, the reductions terminate (L:18). Therefore, $\Psi(\bar{x}_2) = (\bigwedge_{z_u \in E}) \wedge {}^c\phi^2(\bar{x}_2)$, where ${}^c\phi^2(\bar{x}_2) = \mathbf{c}_4$. Note that $(x_1 \dot{\vee} \bar{x}_3) = \mathbf{c}_4$ for ${}^c\phi(\bar{x}_2)$, while $(x_1 \dot{\vee} \bar{x}_3) = \mathbf{c}_1$ for $\phi(\bar{x}_2)$. Note also that ${}^c\phi^2(\bar{x}_2)$ is a sub-formula of $\phi(\bar{x}_2)$ that is *not* reduced from some 3-literal clause in $\phi(\bar{x}_2)$. Because ${}^c\phi^3(\bar{x}_2)$ is empty, $\Psi(\bar{x}_2)$ covers $\phi(\bar{x}_2)$. Also, because $\Psi(\bar{x}_2)$ is satisfiable, ϕ is satisfiable (Corollary 3.11), which has arisen *without* any cycle between the execution of Scan and Discard. This verification arises in this cycle in general, as demonstrated in the sequel.

On the other hand, if ${}^c\phi^3(\bar{x}_2)$ were not empty and the reductions terminated due to $E = E'$ (L:18), then either \bar{x}_2 would become incompatible in Scan \mathcal{N}^φ due to $\Psi(\bar{x}_2) = \perp$ (L:19), or \bar{x}_2 would not yet become incompatible due to $\Psi(\bar{x}_2) = \mathbf{T}$. In the former, Discard (\bar{x}_2) would be executed. In the latter, incompatibility of another $z_i \in \bullet \ell_i$ would be checked (L:4-8 in Scan).

Recall that the order of incompatibility check is insignificant. Assume Incompatible (x_1) is executed *first* in Scan \mathcal{N}^φ . Recall also that $\mathcal{N}^\varphi = \mathcal{N}_1^\varphi$.

Incompatible (x_1) in Scan \mathcal{N}^φ ; $\phi(x_1) = \phi \wedge x_1$: $E \leftarrow \{x_1\}$ (L:1), $\mathfrak{C}^{x_1} = \{1, 2\}$ (L:4), $E_1 \leftarrow \{x_3\}$ (L:6), $E \leftarrow \{x_1, x_3\}$, $\mathbf{c}_1^\bullet \leftarrow \emptyset$ (L:8), and $E_2 \leftarrow \{x_2, \bar{x}_3\}$ (L:6), $E \leftarrow \{x_1, x_3, x_2, \bar{x}_3\}$, $\mathbf{c}_2^\bullet \leftarrow \emptyset$ (L:8). Because $\{x_3, \bar{x}_3\} \subset E$ (L:10), x_1 becomes incompatible for \mathcal{N}^φ , and Discard (x_1) is executed in Scan \mathcal{N}^φ (L:6).

Discard (x_1) in $\text{Scan } \mathcal{N}^\varphi$: $N \leftarrow (N \cup \{\bar{x}_1\})$ (L:1), i.e., $N = \{\bar{x}_1\}$. $\mathfrak{C}^{\bar{x}_1} = \emptyset$ (L:2) and ϕ is not unsatisfiable (L:8). Then, $\mathfrak{C}^{x_1} = \{1, 2\}$ (L:9) over $\phi = (x_1 \dot{\vee} \bar{x}_3) \wedge (x_1 \dot{\vee} \bar{x}_2 \dot{\vee} x_3) \wedge (x_2 \dot{\vee} \bar{x}_3)$, $\mathfrak{c}_1^\bullet \leftarrow (\mathfrak{c}_1^\bullet - \{x_{11}\})$ (L:10), and because $\mathfrak{c}_1^\bullet = \{\bar{x}_{31}\}$, $N \leftarrow (N \cup \{\bar{x}_3\})$ and $\mathfrak{c}_1^\bullet \leftarrow \emptyset$ (L:11), i.e., $N = \{\bar{x}_1, \bar{x}_3\}$. Also, $\mathfrak{c}_2^\bullet \leftarrow (\mathfrak{c}_2^\bullet - \{x_{12}\})$ (L:10), i.e., $\mathfrak{c}_2^\bullet = \{\bar{x}_{22}, x_{32}\}$. Therefore, $\bullet \ell_1 \leftarrow \{\bar{x}_1\}$ (L:13), and $\phi_2 = (\bar{x}_3) \wedge (\bar{x}_2 \oplus x_3) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge \bar{x}_1$ (L:14). As a result, $\mathfrak{C} = \{2, 3\}$, and $C_2 = \{\bar{x}_{22}, x_{32}\}$ and $C_3 = \{x_{23}, \bar{x}_{33}\}$ in \mathcal{N}_2^φ . Note that $\mathfrak{c}_1^\bullet = \{\bar{x}_{31}\}$, $|\bullet \ell_3| = 2$ and $\mathfrak{c}_4^\bullet = \{\bar{x}_{14}\}$, $|\bullet \ell_1| = 1$ in $\mathcal{N}_2^\varphi / \phi_2$. Consequently, $\text{Scan } \mathcal{N}_2^\varphi$ is executed.

$\text{Scan } \mathcal{N}_2^\varphi$: Because $\mathfrak{c}_1^\bullet = \{\bar{x}_{31}\}$ and $|\bullet \ell_3| = 2$ (L:1), i.e., \bar{x}_3 is necessary for $\phi_2 = \bar{x}_3 \wedge (\bar{x}_2 \oplus x_3) \wedge (x_2 \dot{\vee} \bar{x}_3) \wedge \bar{x}_1$, **Discard** (x_3) is executed (L:2).

Discard (x_3) on \mathcal{N}_2^φ : $N \leftarrow (N \cup \{\bar{x}_3\})$ (L:1); $N \leftarrow (\{\bar{x}_1, \bar{x}_3\} \cup \{\bar{x}_3\})$. $\mathfrak{C}^{\bar{x}_3} = \{3\}$ over ϕ_2 (L:2), $N_3 \leftarrow \{\bar{x}_2\}$ (L:3-5), $N \leftarrow (\{\bar{x}_1, \bar{x}_3\} \cup N_3)$, $\mathfrak{c}_3^\bullet \leftarrow \emptyset$ (L:6), and $\phi_2 \equiv \phi$ is not unsatisfiable over $N = \{\bar{x}_1, \bar{x}_3, \bar{x}_2\}$ (L:8). $\mathfrak{C}^{x_3} = \{2\}$ over ϕ_2 (L:9), $\mathfrak{c}_2^\bullet \leftarrow (\mathfrak{c}_2^\bullet - \{x_{32}\})$ (L:10), i.e., $\mathfrak{c}_2^\bullet \leftarrow \{\bar{x}_{22}\}$, and $N \leftarrow (N \cup \{\bar{x}_2\})$, $\mathfrak{c}_2^\bullet \leftarrow \emptyset$ (L:11). As a result, $N = \{\bar{x}_1, \bar{x}_3, \bar{x}_2\}$. Therefore, $\bullet \ell_3 \leftarrow \{\bar{x}_3\}$ (L:13), and $\phi_3 = \bar{x}_1 \wedge \bar{x}_3 \wedge \bar{x}_2$ (L:14). Consequently, $\text{Scan } \mathcal{N}_3^\varphi$ is executed.

$\text{Scan } \mathcal{N}_3^\varphi$: Because $\mathfrak{c}_3^\bullet = \{\bar{x}_{23}\}$ and $|\bullet \ell_2| = 2$ in $\mathcal{N}_3^\varphi / \phi_3$ (L:1), i.e., \bar{x}_2 is necessary for $\phi_3 = \bar{x}_1 \wedge \bar{x}_3 \wedge \bar{x}_2$, **Discard** (x_2) is executed (L:2). Note that $\mathfrak{c}_1^\bullet = \{\bar{x}_{11}\}$, $|\bullet \ell_1| = 1$ and $\mathfrak{c}_2^\bullet = \{\bar{x}_{32}\}$, $|\bullet \ell_3| = 1$ in $\mathcal{N}_3^\varphi / \phi_3$.

Discard (x_2) in \mathcal{N}_3^φ : $\mathfrak{C} = \{k : |\mathfrak{c}_k^\bullet| > 1\} = \emptyset$ over ϕ_3 . Then, $\mathfrak{C}^{\bar{x}_2} = \emptyset$ and $\mathfrak{C}^{x_2} = \emptyset$. Note that $\mathfrak{C}^{\bar{x}_2} = \{k \in \mathfrak{C} : \bar{x}_{2k} \Rightarrow \bar{x}_2\}$. Therefore, $\bullet \ell_2 \leftarrow \{\bar{x}_2\}$, and $\phi_4 = \phi_3$. Consequently, $\text{Scan } \mathcal{N}_4^\varphi$ is executed.

$\text{Scan } \mathcal{N}_4^\varphi$: Because $|\bullet \ell_i| = 1 \ \forall i \in \mathfrak{L}$ (L:1/4 in Scan), $\{\bullet \ell_1, \bullet \ell_2, \bullet \ell_3\}$ satisfies $\phi \equiv \phi_4 = \bar{x}_1 \wedge \bar{x}_3 \wedge \bar{x}_2$, where $\bullet \ell_i = \{\bar{x}_i\} \ \forall i \in \{1, 2, 3\}$.

4 Efficiency of the \mathcal{N}^φ scan

Complexity of the algorithm **Scan** is determined as follows. z_v is incompatible iff $\Psi_s(z_v) = \perp$ (Theorem 3.10). Because $\Psi_s(z_v)$ is of 2SAT/XOR-SAT formula, its complexity is assumed to be n^3 , where $n = |P_2|$ is the number of the literals. $\Psi_s(z_v)$ is determined when the reductions of the clauses terminate (L:18 in **Incompatible** (z_v)). Then, the number of the reductions is the number of the clauses, i.e., $|P_0| = m$, where $P_0 = \{\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_m\}$ denotes the clauses. Thus, the complexity of **Incompatible** (z_v) is $m + n^3$. The number of the incompatibility checks is $|T_2| = 2|P_2| = 2n$ (L:4-8 in **Scan**), where $P_2 = \{\ell_1, \ell_2, \dots, \ell_n\}$ denotes the literals. Also, \mathcal{N}^φ is re-scanned each time some z_v is discarded, and the number of the re-scans is $|P_2| = n$. Therefore, the complexity of **Scan** is $(m + n^3) \times 2n \times n = 2mn^2 + 2n^5$, i.e., $O(n^5)$.

On the other hand, the complexity of **Scan** becomes the complexity of **Incompatible** (z_v) , i.e., $O(n^3)$, if Corollary 3.11 holds. Also, efficiency of **Scan** is improved by finding incompatible literals promptly, because incompatible literals facilitate checking un/satisfiability via the reductions. Thus, checking incompatibility of z_1, z_2, \dots, z_n in parallel improves the efficiency.

5 Conclusion

Reachability in safe acyclic PNs proves to be effective to attack the \mathcal{P} vs \mathcal{NP} problem, because some 2SAT/XOR-SAT formula arisen in the *inversed* PN checks if the truth assignment of a literal (transition firing) z_v is incompatible for the satisfiability of the 3SAT formula (the reachability of the target state in the inversed PN). If z_v is incompatible, then z_v is false, i.e., z_v is discarded and \bar{z}_v becomes true. This incompatibility reduces, by means of exactly-1 disjunction $\dot{\vee}$, a clause $(\bar{z}_v \dot{\vee} z_i \dot{\vee} z_j)$ to the conjunction $(\bar{z}_v \wedge \bar{z}_i \wedge \bar{z}_j)$, and a 3-literal clause $(z_v \dot{\vee} z_u \dot{\vee} z_x)$ to the 2-literal clause $(z_u \oplus z_x)$. The Exactly-1 3SAT formulation is facilitated by sets of conflicts in the PN, i.e., a clause corresponds to a set of conflicts. Checking incompatibility in parallel is possible also, which further improves the efficiency. Because the complexity of checking un/satisfiability is $O(n^5)$, it is the case that $\mathcal{P} = \mathcal{NP} = \text{co}\mathcal{NP}$.

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